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Integrability of the one-dimensional Bariev model

Masahiro Shiroishi and Miki Wadati

Department of Physics, Faculty of Science, University of Tokyo, Hongo 7-3-1, Bunkyo-ku, Tokyo 113, Japan

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Abstract. We investigate the exact integrability of the one-dimensional (1D) Bariev model in the framework of the quantum inverse scattering method (QISM). Using the Jordan–Wigner transformation, the 1D Bariev model can be regarded as a coupled spin model. We construct the higher conserved currents which commute with the Hamiltonian. The explicit form of the conserved currents helps us to infer the L-operator of the 1D Bariev model. From the L-operator, we construct a transfer matrix which is a generating function of the conserved currents. We also find the corresponding R-matrix which satisfies the Yang–Baxter relation. Thus the exact integrability of the 1D Bariev model is established. The R-matrix does not have the 'difference property' for the spectral parameter, as in the case of the 1D Hubbard model. We also provide the Lax representation and the fermionic formulation of the Yang–Baxter relation.

1. Introduction

Recently, there has been much interest in the strongly correlated electron systems in relation to high-T_c superconductivity. Several models have been proposed. Among them, some are known to be exactly solvable in one dimension by the coordinate Bethe ansatz [1–4]. The one-dimensional (1D) Hubbard model and the supersymmetric t-J model are the most famous ones and have been investigated quite rigorously. Based on the Bethe ansatz equation, we can extract many physical properties of the models (see the reprint volume [5]). For example, it is possible to obtain the low-energy gapless excitation spectrum around the ground state by the finite-size scaling method [6, 7]. It exhibits the universal long-distance properties of these models which are characterized as a Tomonaga– Luttinger liquid. The critical exponents of the correlation functions can also be evaluated by use of the predictions of conformal field theory (CFT) [8–10].

On the other hand, the quantum inverse scattering method (QISM) is the most powerful method to treat the 1D exactly solvable models [11–15]. The QISM allows us to show the existence of an infinite number of conserved currents and the diagonalization of the transfer matrix. It is also possible to derive the explicit expressions for the correlation functions if we can apply the algebraic Bethe ansatz [15]. With this approach, we can say that the critical exponents of the correlation functions depend only on the underlying R-matrix which satisfies the Yang–Baxter equation [15]. Thus it is desirable to investigate the 1D exactly solvable models from the point of view of the QISM.

For the supersymmetric t-J model, the QISM has been successfully applied. The higher conservation laws are established, and the transfer matrix is diagonalized by means of the algebraic Bethe ansatz [16, 17]. For the 1D Hubbard model, the situation is not so

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conclusive. The machinary of the QISM was only partially applied. Shastry introduced the Jordan–Wigner transformation to change the model into the coupled XY model and found some non-trivial higher conserved currents [18]. From the form of the higher conserved currents, the corresponding two-dimensional (2D) classical statistical model was inferred [18]. The integrability of the model was proved by showing the Yang–Baxter relation, which assures the existence of an infinite number of conserved currents [19–22]. However, because of the complexity of the R-matrix, the diagonalization of the transfer matrix is a very difficult problem [20]. The spectral parameter of the R-matrix does not have the 'difference property' (3.3) (see later), which makes, for example, the conservation laws very complicated. In this sense, the 1D Hubbard model has been considered as an exceptional model among the exactly solvable models. We note that the algebraic Bethe ansatz used to obtain the eigenvalues of the transfer matrix has recently been reported by Ramos and Martins [24].

Another type of 1D highly correlated electron system was proposed and solved by Bariev (1D Bariev model) [3]. The Hamiltonian does not have a Coulomb interaction term, but has the bond-charge interaction which makes the hopping of the electrons correlated. It resembles the Hirsch's hole superconductivity model [24, 25]. The spin excitations of the 1D Bariev model have a gap and only the charge excitations are gapless. From the finite-size scaling analysis, it was shown that the 1D Bariev model in the attractive case actually has a tendency of the superconductivity [25]. That is, there is a region where the correlation of the superconducting singlet pairs is dominant over the density–density correlation [25]. For the above reasons, the 1D Bariev model and its related models have attracted much attention [27–29].

The 1D Bariev model can also be regarded as the coupled XY model by means of the Jordan–Wigner transformation, which allow us to put the model in the framework of the QISM. As a coupled spin model, the 1D Bariev model is nothing but a linear combination of the two non-interactive XY models and the generalized XY model [30].

The aim of this paper is to investigate the 1D Bariev model in the framework of the QISM. Since the 1D Bariev model has an interpretation of the coupled XY model, we can apply the method developed in the case of the 1D Hubbard model. This approach for the 1D Bariev model was studied by Zhou to some extent [31]. We first discuss the higher conserved currents of the 1D Bariev model in detail. From the explicit form of the higher conserved currents, we can assume the L-operator and the transfer matrix. We propose a different L-operator from Zhou's. We have also found the corresponding R-matrix, which fulfills the Yang–Baxter relation. Thus the exact integrability of the 1D Bariev model is established. The obtained R-matrix enjoys some properties in common with the R-matrix for the 1D Hubbard model. In particular, the R-matrix does not have the 'difference property' (3.3) for the spectral parameter. Moreover, the constraints among the spectral parameters take a very similar form to the 1D Hubbard model. Our results may provide a basis for the further study of the 1D Bariev model.

The outline of this paper is as follows. In section 2, we introduce the 1D Bariev model and its equivalent spin model. In section 3, we first discuss a peculiarity of the Hamiltonian which prevents the recursive construction of the higher conserved currents. Then we adopt a more direct method to find the higher conserved currents. In section 4, we propose a new L-operator. The commutativity of the transfer matrix and the Hamiltonian is proved by means of the Sutherland equation. In section 5, we solve the Yang–Baxter relation to obtain the explicit expression of the R-matrix. We also discuss some fundamental properties of the R-matrix. In section 6, we present the Lax representation of the 1D Bariev model which follows from the Yang–Baxter relation. In section 7, we formulate the Yang–Baxter relation in a fermionic fashion, by use of the Jordan–Wigner transformation. The last section is devoted to concluding remarks. The Yang–Baxter equation for the R-matrix is conjectured.

2. The 1D Bariev model

Let $c_{ns}^\dagger$ and c_{ns} denote fermionic creation and annihilation operators with spins $s(\uparrow$ or $\downarrow)$ at site n $(n = 1, 2, \ldots, N)$. They satisfy the anti-commutation relations

$$\{c_{ns}, c_{n's'}\} = \{c_{ns}^\dagger, c_{n's'}^\dagger\} = 0 \qquad \{c_{ns}^\dagger, c_{n's'}\} = \delta_{nn'}\delta_{ss'}. \tag{2.1}$$

The Hamiltonian of the 1D Bariev model [3] is

$$H_{\text{Bariev}} = -\sum_n [(c_{n+1\uparrow}^\dagger c_{n\uparrow} + c_{n\uparrow}^\dagger c_{n+1\uparrow}) \exp(\eta n_{n+1\downarrow}) + (c_{n\downarrow}^\dagger c_{n+1\downarrow} + c_{n+1\downarrow}^\dagger c_{n\downarrow}) \exp(\eta n_{n\uparrow})] \tag{2.2}$$

where η is the coupling constant describing the correlated hopping and n_{ns} is the number density operator

$$n_{ns} = c_{ns}^\dagger c_{ns}. \tag{2.3}$$

We introduce the Jordan–Wigner transformation which connects the fermion operators and the spin operators,

$$\begin{aligned} c_{n\uparrow} &= \exp\left(\mathrm{i}\pi \sum_{l=1}^{n-1}(n_{l\uparrow} - 1)\right)\sigma_n^- \\ c_{n\downarrow} &= \exp\left(\mathrm{i}\pi \sum_{l=1}^{n-1}(n_{l\downarrow} - 1)\right) \exp\left(i\pi \sum_{l=1}^{N}(n_{l\uparrow} - 1)\right)\tau_n^-. \end{aligned} \tag{2.4}$$

Here σ and τ are two species of the Pauli matrices and commute each other. Applying the Jordan–Wigner transformation (2.4) to the Hamiltonian (2.2), and multiplying an overall factor, we obtain the following coupled spin model,

$$H = \sum_n H_{n+1,n} = \sum_n [(\sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+)(1 + U\tau_{n+1}^z) + (\tau_n^+\tau_{n+1}^- + \tau_n^-\tau_{n+1}^+)(1 + U\sigma_n^z)] \tag{2.5}$$

where

$$U = \tanh\frac{\eta}{2}. \tag{2.6}$$

3. Conserved currents

We investigate the exact integrability of the 1D Bariev model. It is a starting point to look for the higher conserved currents, which commute with the Hamiltonian. One readily verifies that the Hamiltonian density $H_{n+1,n}$ does not satisfy the so-called Reshetikhin's criterion [13] unless $U = 0$, or ∞. That is, if U is a non-zero finite constant, the double commutator

$$[H_{12} + H_{23}, [H_{12}, H_{23}]] \tag{3.1}$$

cannot be expressed as

$$X_{12} - X_{23} \tag{3.2}$$

where X_{ij} is an operator which depends only on the space i and j. The Reshetikhin's criterion follows from the assumption that the R-matrix of the model has the 'difference property' for the spectral parameter,

$$\check{R}(u, v) = \check{R}(u - v). \tag{3.3}$$

This fact implies that the R-matrix for the 1D Bariev model does not have the 'difference property' (3.3) as in the case of the 1D Hubbard model. If the R-matrix has the 'difference property' and the Reshetikhin's criterion is satisfied, the higher conserved currents are recursively produced by taking the commutator with the boost operator [32, 33],

$$I^{(n+1)} = [B, I^{(n)}]. \tag{3.4}$$

Here

$$B = \sum_n n H_{n+1,n} \tag{3.5}$$

and $I^{(n)}$ is the nth conserved current derived by the logarithmic derivative of the transfer matrix. Since the Hamiltonian density of the Bariev model does not satisfy the Reshetikhin's criterion, we should resort to a more direct method to obtain the higher conserved currents [20, 34, 35].

The Hamiltonian of the 1D Bariev model can be written as

$$H = H_0 + U H_1 \tag{3.6}$$

where

$$H_0 = \sum_n (\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+) + (\tau_n^+ \tau_{n+1}^- + \tau_n^- \tau_{n+1}^+)$$
$$H_1 = \sum_n (\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+)\tau_{n+1}^z + \sigma_n^z(\tau_n^+ \tau_{n+1}^- + \tau_n^- \tau_{n+1}^+).$$

We note that the H_0 (H_1) correspond to the $U = 0$ ($U = \infty$) limit of the 1D Bariev model. The Hamiltonian H_1 is a special case of the generalized XY model found by Suzuki [30]. Now we choose the form of the higher conserved currents as follows,

$$J = J_0 + U J_1 + U^2 J_2 \tag{3.7}$$
$$K = K_0 + U K_1 + U^2 K_2 + U^3 K_3 \tag{3.8}$$

where we assume that the current J contains at most 3-spin interactions, and K, 4-spin interactions. The explicit form of J_i is determined by the commutativity with the Hamiltonian,

$$[H, J] = 0 \tag{3.9}$$

which is equivalent to the following set of equations,

$$[H_0, J_0] = [H_1, J_2] = 0 \tag{3.10}$$
$$[H_1, J_0] + [H_0, J_1] = 0 \tag{3.11}$$
$$[H_0, J_2] + [H_1, J_1] = 0. \tag{3.12}$$

Since the Reshetikhin's criterion is satisfied in the cases $U = 0$ and $U = \infty$, the explicit forms of J_0 and J_2 are calculated by using the boost operator approach,

$$J_0 = \sum_n [(\sigma_{n-1}^+ \sigma_n^z \sigma_{n+1}^- - \sigma_{n-1}^- \sigma_n^z \sigma_{n+1}^+) + (\tau_{n-1}^+ \tau_n^z \tau_{n+1}^- - \tau_{n-1}^- \tau_n^z \tau_{n+1}^+)]$$
$$J_2 = \sum_n \Big[(\sigma_{n-1}^+ \sigma_n^z \sigma_{n+1}^- - \sigma_{n-1}^- \sigma_n^z \sigma_{n+1}^+)\tau_n^z \tau_{n+1}^z + \sigma_{n-1}^z \sigma_n^z (\tau_{n-1}^+ \tau_n^z \tau_{n+1}^- - \tau_{n-1}^- \tau_n^z \tau_{n+1}^+)]. \tag{3.13}$$

Substituting J_0 and J_2 into (3.11) and (3.12), we find

$$\begin{aligned}J_1 = \sum_n [&(\sigma^+_{n-1}\sigma^z_n\sigma^-_{n+1} - \sigma^-_{n-1}\sigma^z_n\sigma^+_{n+1})(\tau^z_n + \tau^z_{n+1}) + (\sigma^z_{n-1} + \sigma^z_n)(\tau^+_{n-1}\tau^z_n\tau^-_{n+1} - \tau^-_{n-1}\tau^z_n\tau^+_{n+1})\\ &+4(\sigma^-_{n-1}\sigma^+_n\tau^-_n\tau^+_{n+1} - \sigma^+_{n-1}\sigma^-_n\tau^+_n\tau^-_{n+1})\\ &+4(\sigma^-_n\sigma^+_{n+1}\tau^-_n\tau^+_{n+1} - \sigma^+_n\sigma^-_{n+1}\tau^+_n\tau^-_{n+1})]\end{aligned} \tag{3.14}$$

which was first found by Zhou [31]. Similarly, we can construct the conserved current K. This time the condition

$$[H, K] = 0 \tag{3.15}$$

is equivalent to

$$[H_0, K_0] = [H_1, K_3] = 0 \tag{3.16}$$

$$[H_1, K_0] + [H_0, K_1] = 0 \tag{3.17}$$

$$[H_1, K_1] + [H_0, K_2] = 0 \tag{3.18}$$

$$[H_1, K_2] + [H_0, K_3] = 0. \tag{3.19}$$

The result is as follows,

$$K_0 = \sum_n [(\sigma^+_{n-1}\sigma^z_n\sigma^z_{n+1}\sigma^-_{n+2} + \sigma^-_{n-1}\sigma^z_n\sigma^z_{n+1}\sigma^+_{n+2}) + (\tau^+_{n-1}\tau^z_n\tau^z_{n+1}\tau^-_{n+2} + \tau^-_{n-1}\tau^z_n\tau^z_{n+1}\tau^+_{n+2})] \tag{3.20}$$

$$\begin{aligned}K_1 = \sum_n [&(\sigma^+_{n-1}\sigma^z_n\sigma^z_{n+1}\sigma^-_{n+2} + \sigma^-_{n-1}\sigma^z_n\sigma^z_{n+1}\sigma^+_{n+2})(\tau^z_n + \tau^z_{n+1} + \tau^z_{n+2})\\ &+(\sigma^z_{n-1} + \sigma^z_n + \sigma^z_{n+1})(\tau^+_{n-1}\tau^z_n\tau^z_{n+1}\tau^-_{n+2} + \tau^-_{n-1}\tau^z_n\tau^z_{n+1}\tau^+_{n+2})\\ &-4\{\sigma^+_{n-1}\sigma^z_n\sigma^-_{n+1}(\tau^+_{n-1}\tau^-_n + \tau^+_n\tau^-_{n+1} + \tau^+_{n+1}\tau^+_{n+2})\\ &+\sigma^-_{n-1}\sigma^z_n\sigma^+_{n+1}(\tau^-_{n-1}\tau^+_n + \tau^-_n\tau^+_{n+1} + \tau^-_{n+1}\tau^+_{n+2})\\ &+(\sigma^+_{n-1}\sigma^-_n + \sigma^+_n\sigma^-_{n+1} + \sigma^+_{n+1}\sigma^-_{n+2})\tau^+_n\tau^z_{n+1}\tau^-_{n+2}\\ &+(\sigma^-_{n-1}\sigma^+_n + \sigma^-_n\sigma^+_{n+1} + \sigma^-_{n+1}\sigma^+_{n+2})\tau^-_n\tau^z_{n+1}\tau^+_{n+2}\}]\end{aligned} \tag{3.21}$$

$$\begin{aligned}K_2 = \sum_n [&(\sigma^+_{n-1}\sigma^z_n\sigma^z_{n+1}\sigma^-_{n+2} + \sigma^-_{n-1}\sigma^z_n\sigma^z_{n+1}\sigma^+_{n+2})(\tau^z_n\tau^z_{n+1} + \tau^z_{n+1}\tau^z_{n+2} + \tau^z_n\tau^z_{n+2})\\ &+(\sigma^z_{n-1}\sigma^z_n + \sigma^z_n\sigma^z_{n+1} + \sigma^z_{n-1}\sigma^z_{n+1})(\tau^+_{n-1}\tau^z_n\tau^z_{n+1}\tau^-_{n+2} + \tau^-_{n-1}\tau^z_n\tau^z_{n+1}\tau^+_{n+2})\\ &-4(\sigma^+_{n-1}\sigma^z_n\sigma^-_{n+1}\tau^+_{n-1}\tau^-_n\tau^z_{n+1} + \sigma^-_{n-1}\sigma^z_n\sigma^+_{n+1}\tau^-_{n-1}\tau^+_n\tau^z_{n+1}\\ &+\sigma^z_{n-1}\sigma^+_n\sigma^-_{n+1}\tau^+_{n-1}\tau^z_n\tau^-_{n+1} + \sigma^z_{n-1}\sigma^-_n\sigma^+_{n+1}\tau^-_{n-1}\tau^z_n\tau^+_{n+1}\\ &+\sigma^+_{n-1}\sigma^-_n\sigma^z_{n+1}\tau^+_n\tau^z_{n+1}\tau^-_{n+2} + \sigma^-_{n-1}\sigma^+_n\sigma^z_{n+1}\tau^-_n\tau^z_{n+1}\tau^+_{n+2}\\ &+\sigma^+_{n-1}\sigma^z_n\sigma^-_{n+1}\tau^z_n\tau^+_{n+1}\tau^-_{n+2} + \sigma^-_{n-1}\sigma^z_n\sigma^+_{n+1}\tau^z_n\tau^-_{n+1}\tau^+_{n+2})\\ &+4(\sigma^+_{n-1}\sigma^-_n\tau^+_{n-1}\tau^-_{n+1} + \sigma^-_{n-1}\sigma^+_n\tau^-_{n-1}\tau^+_{n+1}\\ &+\sigma^+_{n-1}\sigma^-_{n+1}\tau^+_n\tau^-_{n+1} + \sigma^-_{n-1}\sigma^+_{n+1}\tau^-_n\tau^+_{n+1})]\end{aligned}$$

$$\begin{aligned}K_3 = \sum_n [&(\sigma^+_{n-1}\sigma^z_n\sigma^z_{n+1}\sigma^-_{n+2} + \sigma^-_{n-1}\sigma^z_n\sigma^z_{n+1}\sigma^+_{n+2})\tau^z_n\tau^z_{n+1}\tau^z_{n+2}\\ &+\sigma^z_{n-1}\sigma^z_n\sigma^z_{n+1}(\tau^+_{n-1}\tau^z_n\tau^z_{n+1}\tau^-_{n+2} + \tau^-_{n-1}\tau^z_n\tau^z_{n+1}\tau^+_{n+2})].\end{aligned} \tag{3.22}$$

The conserved currents are supposed to be involutive. In fact, we have verified that

$$[J, K] = 0 \tag{3.23}$$

which now is equivalent to

$$[J_0, K_0] = [J_2, K_3] = 0 \tag{3.24}$$

$$[J_0, K_1] + [J_1, K_0] = 0 \tag{3.25}$$

$$[J_0, K_2] + [J_1, K_1] + [J_2, K_0] = 0 \tag{3.26}$$

$$[J_0, K_3] + [J_1, K_2] + [J_2, K_1] = 0 \tag{3.27}$$
$$[J_1, K_3] + [J_2, K_2] = 0. \tag{3.28}$$

4. *L*-operator

In this section, we discuss the L-operator and the transfer matrix for the 1D Bariev model. The transfer matrix is written in the standard form,

$$T(u) = \mathrm{tr}_a \overleftarrow{\prod_n} L_{n,a}(u) = \mathrm{tr}_a[L_{N,a}(u)L_{N-1,a}(u)\ldots L_{1,a}(u)] \tag{4.1}$$

where a is the auxiliary variable corresponding to the horizontal arrows in the row-to-row transfer matrix. We want to find a set of transfer matrices with a different value of the spectral parameter which commute mutually,

$$[T(u), T(v)] = 0. \tag{4.2}$$

Usually, the commutativity of the transfer matrix is guaranteed by the existence of the R-matrix which satisfies the (local) Yang–Baxter relation [11–15, 36],

$$\check{R}_{12}(u, v)[L_n(u) \otimes L_n(v)] = [L_n(v) \otimes L_n(u)]\check{R}_{12}(u, v). \tag{4.3}$$

First, we discuss the two special cases ($U = 0, \infty$). In these cases, the Yang–Baxter structures are already known.

(1) $U = 0$. The Hamiltonian decouples into the two non-interactive XY models, and the L-operator is

$$L_{n,a}^{(U=0)}(\theta) = L_{n,a}^{(\sigma)}(\theta)L_{n,a}^{(\tau)}(\theta) \tag{4.4}$$

where

$$\begin{aligned} L_{n,a}^{(\sigma)}(\theta) &= \tfrac{1}{2}\cos\theta(1 + \sigma_n^z\sigma_a^z) + \tfrac{1}{2}\sin\theta(1 - \sigma_n^z\sigma_a^z) + (\sigma_n^+\sigma_a^- + \sigma_a^+\sigma_n^-) \\ L_{n,a}^{(\tau)}(\theta) &= \tfrac{1}{2}\cos\theta(1 + \tau_n^z\tau_a^z) + \tfrac{1}{2}\sin\theta(1 - \tau_n^z\tau_a^z) + (\tau_n^+\tau_a^- + \tau_a^+\tau_n^-). \end{aligned} \tag{4.5}$$

Correspondingly, the R-matrix is given by

$$\check{R}_{12}^{(U=0)}(\theta_1 - \theta_2) = \check{R}_{12}^{(\sigma)}(\theta_1 - \theta_2)\check{R}_{12}^{(\tau)}(\theta_1 - \theta_2) \tag{4.6}$$

where

$$\begin{aligned} \check{R}_{12}^{(\sigma)}(\theta_1 - \theta_2) &= \tfrac{1}{2}\cos(\theta_1 - \theta_2)(1 + \sigma_1^z\sigma_2^z) + \sin(\theta_1 - \theta_2)(\sigma_1^+\sigma_2^- + \sigma_1^+\sigma_2^-) + \tfrac{1}{2}(1 - \sigma_1^z\sigma_2^z) \\ \check{R}_{12}^{(\tau)}(\theta_1 - \theta_2) &= \tfrac{1}{2}\cos(\theta_1 - \theta_2)(1 + \tau_1^z\tau_2^z) + \sin(\theta_1 - \theta_2)(\tau_1^+\tau_2^- + \tau_1^+\tau_2^-) + \tfrac{1}{2}(1 - \tau_1^z\tau_2^z). \end{aligned} \tag{4.7}$$

(2) $U = \infty$. As noted in section 3, the 1D Bariev model reduces to Suzuki's generalized XY model [30] in the $U = \infty$ limit, and the Yang–Baxter relation for this model was found by Lopez [37]. The L-operator is

$$\begin{aligned} L_{n,a}^{(U=\infty)}(\psi) = {} & \{\tfrac{1}{2}\cos\psi(1 + \sigma_n^z\sigma_a^z) + \tfrac{1}{2}\sin\psi(1 - \sigma_n^z\sigma_a^z)\tau_a^z + (\sigma_n^+\sigma_a^- + \sigma_a^+\sigma_n^-)\} \\ & \times\{\tfrac{1}{2}\cos\psi(1 + \tau_n^z\tau_a^z) + \tfrac{1}{2}\sin\psi(1 - \tau_n^z\tau_a^z)\sigma_a^z + (\tau_n^+\tau_a^- + \tau_a^+\tau_n^-)\}. \end{aligned} \tag{4.8}$$

The corresponding R-matrix is given by

$$\check{R}_{12}^{(U=\infty)}(\psi_1 - \psi_2) = F_{12}\{\check{R}_{12}^{(\sigma)}(\psi_1 - \psi_2)\check{R}_{12}^{(\tau)}(\psi_1 - \psi_2)\}F_{12}^{-1} \tag{4.9}$$

where

$$F_{12} \equiv \tfrac{1}{2}(1 + \sigma_1^z + \tau_2^z - \sigma_1^z\tau_2^z). \tag{4.10}$$

We note some useful relations for the operator F_{12},

$$F_{12}^2 = 1$$
$$F_{12}\{(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+) + (\tau_1^+\tau_2^- + \tau_1^-\tau_2^+)\}F_{12}^{-1} = (\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+)\tau_2^z + \sigma_1^z(\tau_1^+\tau_2^- + \tau_1^-\tau_2^+). \tag{4.11}$$

The R matrix (4.9) is related to the trigonometric limit of the quantum Clifford–Hopf algebra $CH_q(4)$ [37].

Now we look for the L-operator for general U. We assume that the L-operator has the following expansion in powers of the spectral parameter u,

$$L_{n,a}(u) = P_{n,a}\left(1 + uH_{n,a} + \frac{u^2}{2!}B_{n,a} + \frac{u^3}{3!}C_{n,a} + \cdots\right) \tag{4.12}$$

where $P_{n,a}$ is the permutation operator. Lüscher [38] has shown that, if the transfer matrix consists of the L-operator (4.1) and the transfer matrices commute mutually, then the nth logarithmic derivative of the transfer matrix,

$$I^{(n)} \equiv \frac{\mathrm{d}^n}{\mathrm{d}u^n}\ln(T^{-1}(0)T(u))\Big|_{u=0} \tag{4.13}$$

is the *local* conserved currents in involution,

$$[I^{(n)}, I^{(m)}] = 0. \tag{4.14}$$

Here the term *local* means that $I^{(n)}$ contains at most $n+1$ spin interactions. Using the expansion of the L-operator (4.12), the explicit forms of the conserved currents are, for instance, given as [20, 39],

$$I^{(1)} = \sum_n H_{n,n-1} \tag{4.15}$$

$$I^{(2)} = \sum_n (B_{n,n-1} - H_{n,n-1}^2) + \sum_n [H_{n+1,n}, H_{n,n-1}] \tag{4.16}$$

$$\begin{aligned}
I^{(3)} = \sum_n &(C_{n,n-1} - H_{n,n-1}^3) \\
&-\frac{3}{2}\sum_n \{H_{n,n-1}(B_{n,n-1} - H_{n,n-1}^2) + (B_{n,n-1} - H_{n,n-1}^2)H_{n,n-1}\} \\
&+\frac{3}{2}\sum_n \{[H_{n+1,n}, B_{n,n-1} - H_{n,n-1}^2] + [B_{n+1,n} - H_{n+1,n}^2, H_{n,n-1}]\} \\
&+\frac{1}{2}\sum_n \{[H_{n+1,n}, [H_{n+1,n}, H_{n,n-1}]] + [[H_{n+1,n}, H_{n,n-1}], H_{n,n-1}]\} \\
&+2\sum_n [[H_{n+1,n}, H_{n,n-1}], H_{n-1,n-2}].
\end{aligned} \tag{4.17}$$

In general, the conserved currents are not determined uniquely, since we can add any lower conserved currents to redefine the higher one. However, if we assume

$$H = I^{(1)} \qquad J = I^{(2)} \qquad K = \tfrac{1}{2}I^{(3)} \tag{4.18}$$

then we can determine the higher terms in (4.12) as follows:

$$\begin{aligned}
B_{n,n-1} &= \tfrac{1}{2}(1 - \sigma_{n-1}^z\sigma_n^z)(1 + U\tau_n^z)^2 + \tfrac{1}{2}(1 - \tau_{n-1}^z\tau_n^z)(1 + U\sigma_{n-1}^z)^2 \\
&\quad +2(\sigma_{n-1}^+\sigma_n^- + \sigma_{n-1}^-\sigma_n^+)(1 + U\sigma_{n-1}^z)(1 + U\tau_n^z)(\tau_{n-1}^+\tau_n^- + \tau_{n-1}^-\tau_n^+) \\
C_{n,n-1} &= \tfrac{3}{2}(\sigma_{n-1}^+\sigma_n^- + \sigma_{n-1}^-\sigma_n^+)(1 + U\sigma_{n-1}^z)^2(1 + U\tau_n)(1 - \tau_{n-1}^z\tau_n^z) \\
&\quad +\tfrac{3}{2}(\tau_{n-1}^+\tau_n^- + \tau_{n-1}^-\tau_n^+)(1 + U\sigma_{n-1}^z)(1 + U\tau_n)^2(1 - \tau_{n-1}^z\tau_n^z).
\end{aligned} \tag{4.19}$$

Based on these results, Zhou [31] found a L-operator for the 1D Bariev model,

$$L_{n,a}(u) = L'_{n,a}(u)L''_{n,a}(u) \tag{4.20}$$

where

$$L'_{n,a}(u) = \tfrac{1}{2}(1+\sigma_n^z\sigma_a^z) + \tfrac{1}{2}(1-\sigma_n^z\sigma_a^z)(1+U\tau_a^z)u + (\sigma_n^+\sigma_a^- + \sigma_n^-\sigma_a^+)\sqrt{1+(1+U\tau_a^z)^2u^2}$$
$$L''_{n,a}(u) = \tfrac{1}{2}(1+\tau_n^z\tau_a^z) + \tfrac{1}{2}(1+U\sigma_a^z)(1-\tau_n^z\tau_a^z)u + \sqrt{1+(1+U\sigma_a^z)^2u^2}(\tau_n^+\tau_a^- + \tau_n^-\tau_a^+). \tag{4.21}$$

The corresponding R-matrix, which satisfies the Yang–Baxter relation

$$\check{R}(u,v)[L_n(u)\otimes L_n(v)] = [L_n(v)\otimes L_n(v)]\check{R}(u,v) \tag{4.22}$$

was also found [31].

There are some disadvantages about the L-operator (4.21).

(1) The L-operator (4.21) does not reduce to (4.4) and (4.8) if we take the limits $U = 0$ and $U = \infty$, respectively. The corresponding R-matrix is not useful when we discuss the difference property (3.3).

(2) It is not easy to formulate the Yang–Baxter relation in a fermionic fashion using the L-operator (4.21).

We shall, therefore, look for another L-operator for the 1D Bariev model which includes (4.4) and (4.8) as special cases. First, we assume the L-operator as

$$L_{n,a} = L^{(1)}_{n,a}L^{(2)}_{n,a} \tag{4.23}$$

where

$$L^{(1)}_{n,a} = \tfrac{1}{2}\cos(\theta+\psi\tau_a^z)(1+\sigma_n^z\sigma_a^z) + \tfrac{1}{2}\sin(\theta+\psi\tau_a^z)(1-\sigma_n^z\sigma_a^z) + (\sigma_n^+\sigma_a^- + \sigma_n^-\sigma_a^+)$$
$$L^{(2)}_{n,a} = \tfrac{1}{2}\cos(\theta+\psi\sigma_a^z)(1+\tau_n^z\tau_a^z) + \tfrac{1}{2}\sin(\theta+\psi\sigma_a^z)(1-\tau_n^z\tau_a^z) + (\tau_n^+\tau_a^- + \tau_n^-\tau_a^+). \tag{4.24}$$

Here θ and ψ are assumed to be spectral parameters. However, as we will see shortly, θ and ψ are not independent. Next, we consider the following relation,

$$[H_{n+1,n}, L_{n+1,a}L_{n,a}] = Q_{n+1,a}L_{n,a} - L_{n+1,a}Q_{n,a} \tag{4.25}$$

where $H_{n+1,n}$ is the Hamiltonian density and $Q_{n,a}$ is some operator. This kind of equation was used to show the commutativity of the 1D quantum spin Hamiltonians and the transfer matrix of the 2D classical vertex models [40, 41, 18]. In fact, if there exists an operator which satisfies (4.25), then the transfer matrix constructed from the L-operator commutes with the Hamiltonian. We call equation (4.25) the Sutherland equation [14]. We assume the operator $Q_{n,a}$ in the form

$$Q_{n,a} = \begin{pmatrix} Q_n^{11} & Q_n^{12}\tau_n^- & Q_n^{13}\sigma_n^- & Q_n^{14}\sigma_n^-\tau_n^- \\ Q_n^{21}\tau_n^+ & Q_n^{22} & Q_n^{23}\sigma_n^-\tau_n^+ & Q_n^{24}\sigma_n^- \\ Q_n^{31}\sigma_n^+ & Q_n^{32}\sigma_n^+\tau_n^- & Q_n^{33} & Q_n^{34}\tau_n^- \\ Q_n^{41}\sigma_n^+\tau_n^+ & Q_n^{42}\sigma_n^+ & Q_n^{43}\tau_n^+ & Q_n^{44} \end{pmatrix} \tag{4.26}$$

where Q_n^{ij}, $(i, j = 1, \ldots, 4)$ are the operators which depend on σ_n^z and τ_n^z. Substituting (4.26) into the Sutherland equation (4.25) we find a constraint between θ and ψ:

$$\frac{\sin 2\psi}{\sin 2\theta} = U. \tag{4.27}$$

Recall that U is the coupling constant of the 1D Bariev model. Under the constraint (4.27), we can obtain Q_n^{ij},

$$
\begin{aligned}
Q_n^{11} &= \tfrac{1}{2}(\sin 2\theta + U\sin 2\psi)(\sigma_n^z + \tau_n^z) - \tfrac{1}{2}(\cos 2\theta + U\cos 2\psi)(1 - \sigma_n^z\tau_n^z)\\
Q_n^{22} &= -\tfrac{1}{2}\cos 2\theta\cos 2\psi(1 + \sigma_n^z\tau_n^z + U(\sigma_n^z + \tau_n^z)) + \tfrac{1}{2}\sin 2\theta\cos 2\psi(\sigma_n^z - \tau_n^z)\\
&\qquad + \tfrac{1}{2}U\sin 2\psi\cos 2\theta(1 - \sigma_n^z\tau_n^z)\\
Q_n^{33} &= -\tfrac{1}{2}\cos 2\theta\cos 2\psi(1 + \sigma_n^z\tau_n^z + U(\sigma_n^z + \tau_n^z)) + \tfrac{1}{2}\sin 2\theta\cos 2\psi(\sigma_n^z - \tau_n^z)\\
&\qquad + \tfrac{1}{2}U\sin 2\psi\cos 2\theta(1 - \sigma_n^z\tau_n^z)\\
Q_n^{44} &= -\tfrac{1}{2}(\sin 2\theta + U\sin 2\psi)(\sigma_n^z + \tau_n^z) - \tfrac{1}{2}(\cos 2\theta - U\cos 2\psi)(1 - \sigma_n^z\tau_n^z)\\
Q_n^{12} &= \tfrac{1}{2}(1-U)\sin(\theta-\psi)(1+\sigma_n^z) - \tfrac{1}{2}(1+U)\cos(\theta-\psi)(1-\sigma_n^z)\\
Q_n^{31} &= \tfrac{1}{2}(1-U)\sin(\theta-\psi)(1+\tau_n^z) + \tfrac{1}{2}(1+U)\cos(\theta-\psi)(1-\tau_n^z)\\
Q_n^{43} &= -\tfrac{1}{2}(1-U)\cos(\theta+\psi)(1+\sigma_n^z) + \tfrac{1}{2}(1+U)\sin(\theta+\psi)(1-\sigma_n^z)\\
Q_n^{24} &= -\tfrac{1}{2}(1-U)\cos(\theta+\psi)(1+\tau_n^z) + \tfrac{1}{2}(1+U)\sin(\theta+\psi)(1-\tau_n^z)\\
Q_n^{21} &= \tfrac{1}{2}(\cos 2\psi - U\cos 2\theta)(\sin(\theta-\psi)(1+\sigma_n^z) - \cos(\theta-\psi)(1-\sigma_n^z))\\
Q_n^{13} &= \tfrac{1}{2}(\cos 2\psi - U\cos 2\theta)(\sin(\theta-\psi)(1+\tau_n^z) - \cos(\theta-\psi)(1-\tau_n^z))\\
Q_n^{34} &= \tfrac{1}{2}(\cos 2\psi + U\cos 2\theta)(-\cos(\theta+\psi)(1+\sigma_n^z) + \sin(\theta+\psi)(1-\sigma_n^z))\\
Q_n^{42} &= \tfrac{1}{2}(\cos 2\psi + U\cos 2\theta)(-\cos(\theta+\psi)(1+\tau_n^z) + \sin(\theta+\psi)(1-\tau_n^z))\\
Q_n^{32} &= -Q_n^{23} = 2\sin 2\psi\\
Q_n^{14} &= Q_n^{41} = 0.
\end{aligned}
\tag{4.28}
$$

Thus we obtain a new L-operator for the 1D Bariev model,

$$
\begin{aligned}
L_{n,a}(\theta) &= \{\tfrac{1}{2}\cos(\theta + \psi\tau_a^z)(1 + \sigma_n^z\sigma_a^z) + \tfrac{1}{2}\sin(\theta + \psi\tau_a^z)(1 - \sigma_n^z\sigma_a^z) + (\sigma_n^+\sigma_a^- + \sigma_n^-\sigma_a^+)\}\\
&\quad\times\{\tfrac{1}{2}\cos(\theta + \psi\sigma_a^z)(1 + \tau_n^z\tau_a^z) + \tfrac{1}{2}\sin(\theta + \psi\sigma_a^z)(1 - \tau_n^z\tau_a^z) + (\tau_n^+\tau_a^- + \tau_n^-\tau_a^+)\}\\
&= \begin{pmatrix}
p_n^+(\theta)q_n^+(\theta) & p_n^+(\theta)\tau_n^- & \sigma_n^-\tilde{q}_n^+(\theta) & \sigma_n^-\tau_n^-\\
\tilde{p}_n^+(\theta)\tau_n^+ & \tilde{p}_n^+(\theta)q_n^-(\theta) & \sigma_n^-\tau_n^+ & \sigma_n^-\tilde{q}_n^-(\theta)\\
\sigma_n^+q_n^+(\theta) & \sigma_n^+\tau_n^- & p_n^-(\theta)\tilde{q}_n^+(\theta) & p_n^-(\theta)\tau_n^-\\
\sigma_n^+\tau_n^+ & \sigma_n^+q_n^-(\theta) & \tilde{p}_n^-(\theta)\tau_n^+ & \tilde{p}_n^-(\theta)\tilde{q}_n^-(\theta)
\end{pmatrix}
\end{aligned}
\tag{4.29}
$$

where

$$
\begin{aligned}
p_n^\pm(\theta) &= \tfrac{1}{2}(\cos(\theta+\psi) + \sin(\theta+\psi)) \pm \tfrac{1}{2}(\cos(\theta+\psi) - \sin(\theta+\psi))\sigma_n^z\\
\tilde{p}_n^\pm(\theta) &= \tfrac{1}{2}(\cos(\theta-\psi) + \sin(\theta-\psi)) \pm \tfrac{1}{2}(\cos(\theta-\psi) - \sin(\theta-\psi))\sigma_n^z\\
q_n^\pm(\theta) &= \tfrac{1}{2}(\cos(\theta+\psi) + \sin(\theta+\psi)) \pm \tfrac{1}{2}(\cos(\theta+\psi) - \sin(\theta+\psi))\tau_n^z\\
\tilde{q}_n^\pm(\theta) &= \tfrac{1}{2}(\cos(\theta-\psi) + \sin(\theta-\psi)) \pm \tfrac{1}{2}(\cos(\theta-\psi) - \sin(\theta-\psi))\tau_n^z.
\end{aligned}
$$

Here the parameter ψ is considered to be a function of θ through the constraint (4.27). Then we write the the Sutherland equation (4.25) as

$$
[H_{n+1,n}, L_{n+1,a}(\theta)L_{n,a}(\theta)] = Q_{n+1,a}(\theta)L_{n,a}(\theta) - L_{n+1,a}(\theta)Q_{n,a}(\theta). \tag{4.30}
$$

The Hamiltonian (2.5) is the logarithmic derivative of the transfer matrix

$$
H = T(0)^{-1}\frac{\mathrm{d}}{\mathrm{d}\theta}T(\theta)\Big|_{\theta=0} \tag{4.31}
$$

where

$$T(\theta) = \mathrm{tr}_a \overleftarrow{\prod_n} L_{n,a}(\theta). \tag{4.32}$$

5. Yang–Baxter relation

In this section, we discuss the Yang–Baxter relation corresponding to the L-operator (4.29). We look for a 16×16 matrix $\check{R}_{12}(\theta_1, \theta_2)$ which satisfies the Yang–Baxter relation

$$\check{R}_{12}(\theta_1, \theta_2)\,[L_n(\theta_1) \otimes L_n(\theta_2)] = [L_n(\theta_2) \otimes L_n(\theta_1)]\,\check{R}_{12}(\theta_1, \theta_2) \tag{5.1}$$

under the constraint

$$\frac{\sin 2\psi_1}{\sin 2\theta_1} = \frac{\sin 2\psi_2}{\sin 2\theta_2} = U. \tag{5.2}$$

Following the method proposed by Olmedilla *et al* [21], the non-zero elements of the R-matrix, $\check{R}_{12}(\theta_1, \theta_2)$, are assumed to be

$$\begin{aligned} &R_{ii} \qquad \text{for } i = 1, 2, \ldots, 16 \\ &R_{2,5},\, R_{5,2},\, R_{3,9},\, R_{9,3},\, R_{4,7},\, R_{7,4},\, R_{4,10},\, R_{10,4},\, R_{4,13} R_{13,4},\, R_{8,14},\, R_{14,8}, \\ &\qquad R_{12,15},\, R_{15,12},\, R_{10,13},\, R_{13,10},\, R_{7,13},\, R_{13,7},\, R_{10,7},\, R_{7,10}. \end{aligned} \tag{5.3}$$

Using the relations

$$\begin{aligned} p_n^{\pm}(\theta)\sigma_n^{\pm} &= \sigma_n^{\pm} p^{\mp}(\theta) = \cos(\theta + \psi)\sigma_n^{\pm} \\ \tilde{p}_n^{\pm}(\theta)\sigma_n^{\pm} &= \sigma_n^{\pm} \tilde{p}^{\mp}(\theta) = \cos(\theta - \psi)\sigma_n^{\pm} \\ \sigma_n^{\pm} p_n^{\pm}(\theta) &= p_n^{\mp}(\theta)\sigma_n^{\pm} = \sin(\theta + \psi)\sigma_n^{\pm} \\ \sigma_n^{\pm} \tilde{p}_n^{\pm}(\theta) &= \tilde{p}_n^{\mp}(\theta)\sigma_n^{\pm} = \sin(\theta - \psi)\sigma_n^{\pm} \end{aligned}$$

and the similar relations for τ spins, we can solve (5.1) to find the non-zero elements of the R-matrix. The result is

$$\check{R}_{12}(\theta_1, \theta_2) = \begin{pmatrix}
\alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_5 & 0 & 0 & \alpha_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_6 & 0 & 0 & 0 & 0 & 0 & \alpha_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_4 & 0 & 0 & \alpha_{18} & 0 & 0 & \alpha_{15} & 0 & 0 & \alpha_{20} & 0 & 0 & 0 \\
0 & \alpha_{12} & 0 & 0 & \alpha_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_{17} & 0 & 0 & \alpha_9 & 0 & 0 & \alpha_{19} & 0 & 0 & \alpha_{18} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_7 & 0 & 0 & 0 & 0 & 0 & \alpha_{14} & 0 & 0 \\
0 & 0 & \alpha_{12} & 0 & 0 & 0 & 0 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_{16} & 0 & 0 & \alpha_{19} & 0 & 0 & \alpha_{10} & 0 & 0 & \alpha_{15} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_8 & 0 & 0 & \alpha_{14} & 0 \\
0 & 0 & 0 & \alpha_{21} & 0 & 0 & \alpha_{17} & 0 & 0 & \alpha_{16} & 0 & 0 & \alpha_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{13} & 0 & 0 & 0 & 0 & 0 & \alpha_8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{13} & 0 & 0 & \alpha_7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_3
\end{pmatrix} \tag{5.4}$$

To describe the non-zero elements of the R-matrix, we introduce the following abbreviations:

$$c_{i\pm} = \cos(\theta_i \pm \psi_i) \qquad s_{i\pm} = \sin(\theta_i \pm \psi_i) \qquad i = 1, 2 \tag{5.5}$$

$$z_{12} = \frac{c_{1+}c_{2-}}{c_{1-}c_{2+}} = \frac{s_{1+}s_{2-}}{s_{1-}s_{2+}}. \tag{5.6}$$

The second equality for z_{12} is due to the constraint (5.2). Then, the non-zero elements of the R-matrix are given by

$$\begin{aligned}
\alpha_1(\theta_1,\theta_2) &= (z_{12}c_{1+}c_{2+} + s_{1+}s_{2+})(c_{1-}c_{2-} + z_{12}^{-1}s_{1-}s_{2-})\\
\alpha_2(\theta_1,\theta_2) &= (c_{1+}c_{2+} + z_{12}s_{1+}s_{2+})(c_{1-}c_{2-} + z_{12}^{-1}s_{1-}s_{2-})\\
\alpha_3(\theta_1,\theta_2) &= (c_{1+}c_{2+} + z_{12}s_{1+}s_{2+})(z_{12}^{-1}c_{1-}c_{2-} + s_{1-}s_{2-})\\
\alpha_4(\theta_1,\theta_2) &= 1\\
\alpha_5(\theta_1,\theta_2) &= c_{1-}c_{2-} + z_{12}^{-1}s_{1-}s_{2-}\\
\alpha_6(\theta_1,\theta_2) &= z_{12}c_{1-}c_{2-} + s_{1-}s_{2-}\\
\alpha_7(\theta_1,\theta_2) &= c_{1+}c_{2+} + z_{12}s_{1+}s_{2+}\\
\alpha_8(\theta_1,\theta_2) &= z_{12}^{-1}c_{1+}c_{2+} + s_{1+}s_{2+}\\
\alpha_9(\theta_1,\theta_2) &= 1 + \frac{c_{1+}}{c_{2+}}\left(\frac{s_{1-}}{s_{2-}} - \frac{c_{1+}}{c_{2+}}\right)(c_{2+}^2 - c_{2-}^2)\\
\alpha_{10}(\theta_1,\theta_2) &= 1 + \frac{c_{1-}}{c_{2-}}\left(\frac{s_{1+}}{s_{2+}} - \frac{c_{1-}}{c_{2-}}\right)(c_{2-}^2 - c_{2+}^2)\\
\alpha_{11}(\theta_1,\theta_2) &= (s_{1+}c_{2+} - z_{12}c_{1+}s_{2+})(c_{1-}c_{2-} + z_{12}^{-1}s_{1-}s_{2-})\\
\alpha_{12}(\theta_1,\theta_2) &= (z_{12}s_{1+}c_{2+} - c_{1+}s_{2+})(c_{1-}c_{2-} + z_{12}^{-1}s_{1-}s_{2-})\\
\alpha_{13}(\theta_1,\theta_2) &= (s_{1-}c_{2-} - z_{12}^{-1}c_{1-}s_{2-})(c_{1+}c_{2+} + z_{12}s_{1+}s_{2+})\\
\alpha_{14}(\theta_1,\theta_2) &= (z_{12}^{-1}s_{1-}c_{2-} - c_{1-}s_{2-})(c_{1+}c_{2+} + z_{12}s_{1+}s_{2+})\\
\alpha_{15}(\theta_1,\theta_2) &= s_{1-}c_{2+} - c_{1+}s_{2-}\\
\alpha_{16}(\theta_1,\theta_2) &= s_{1-}c_{2+} - \frac{s_{1-}c_{2+}}{s_{1+}c_{2-}}c_{1-}s_{2+}\\
\alpha_{17}(\theta_1,\theta_2) &= s_{1+}c_{2-} - c_{1-}s_{2+}\\
\alpha_{18}(\theta_1,\theta_2) &= s_{1+}c_{2-} - \frac{s_{1+}c_{2-}}{s_{1-}c_{2+}}c_{1+}s_{2-}\\
\alpha_{19}(\theta_1,\theta_2) &= (s_{1+}c_{2-} - c_{1-}s_{2+})(s_{1-}c_{2+} - c_{1+}s_{2-})\\
\alpha_{20}(\theta_1,\theta_2) &= (s_{1-}c_{2+} - c_{1+}s_{2-})\left(s_{1-}c_{2+} - \frac{s_{1+}c_{2-}}{s_{1-}c_{2+}}c_{1-}s_{2+}\right)\\
\alpha_{21}(\theta_1,\theta_2) &= (s_{1+}c_{2-} - c_{1-}s_{2+})\left(s_{1+}c_{2-} - \frac{s_{1-}c_{2+}}{s_{1+}c_{2-}}c_{1+}s_{2-}\right).
\end{aligned} \tag{5.7}$$

We have normalized the R-matrix so that it satisfies the initial condition

$$\check{R}(\theta_1 = \theta_0, \theta_2 = \theta_0) = \mathrm{id} \tag{5.8}$$

where id is the identity operator.

Now we shall observe the $U = 0$ and $U = \infty$ limit of the R-matrix (5.4).

(1) $U = 0$. The parameters ψ_i, $i = 1, 2$ are zero and z_{12} is unity, and therefore we have

$$\check{R}_{12}(\theta_1,\theta_2) \longrightarrow \check{R}_{12}^{(U=0)}(\theta_1 - \theta_2) = \check{R}_{12}^{(\sigma)}(\theta_1 - \theta_2)\check{R}_{12}^{(\tau)}(\theta_1 - \theta_2). \tag{5.9}$$

(2) $U = \infty$. The spectral parameter θ_i, $i = 1, 2$, are zero, and z_{12} is unity. We consider ψ_i, $i = 1, 2$, as the spectral parameters. Then

$$\check{R}_{12}(\theta_1,\theta_2) \longrightarrow \check{R}_{12}^{(U=\infty)}(\psi_1 - \psi_2) = F_{12}\{\check{R}_{12}^{(\sigma)}(\psi_1 - \psi_2)\check{R}_{12}^{(\tau)}(\psi_1 - \psi_2)\}F_{12}^{-1}. \tag{5.10}$$

Thus the R-matrix (5.4) reduces to the one discussed in section 4 in the limit $U = 0$ and $U = \infty$.

For the later use, we introduce another equivalent form of the Yang–Baxter relation (5.1),

$$R_{12}(\theta_1, \theta_2) L_{n,1}(\theta_1) L_{n,2}(\theta_2) = L_{n,2}(\theta_2) L_{n,1}(\theta_1) R_{12}(\theta_1, \theta_2) \tag{5.11}$$

where, with P_{12} being the permutation operator,

$$R_{12}(\theta_1, \theta_2) \equiv P_{12} \check{R}_{12}(\theta_1, \theta_2). \tag{5.12}$$

The R-matrix fulfills the following unitarity relation,

$$\check{R}_{12}(\theta_2, \theta_1) \check{R}_{12}(\theta_1, \theta_2) = \rho(\theta_1, \theta_2)\,\mathrm{id} \tag{5.13}$$

or equivalently

$$R_{21}(\theta_2, \theta_1) R_{12}(\theta_1, \theta_2) = \rho(\theta_1, \theta_2)\,\mathrm{id} \tag{5.14}$$

where

$$\begin{aligned}\rho(\theta_1, \theta_2) = {} & (z_{12} c_{1+} c_{2+} + s_{1+} s_{2+})(c_{1-} c_{2-} + z_{12}^{-1} s_{1-} s_{2-}) \\ & \times (z_{12}^{-1} c_{1+} c_{2+} + s_{1+} s_{2+})(c_{1-} c_{2-} + z_{12} s_{1-} s_{2-}).\end{aligned} \tag{5.15}$$

Now we show that the Sutherland equation (4.30) is derived from the Yang–Baxter relation. We note

$$L_{n,a}(\theta) = R_{a,n}(\theta, 0). \tag{5.16}$$

By differentiating the Yang–Baxter relation (5.11) with respect to θ_2 at $\theta_2 = 0$, we obtain

$$\begin{aligned}& \frac{\mathrm{d}}{\mathrm{d}\theta_2} R_{1,n}(\theta_1, \theta_2)\bigg|_{\theta_2=0} L_{2,1}(\theta_1) + L_{n,1}(\theta_1) L_{2,1}(\theta_1) H_{n,2} \\ & \qquad = H_{n,2} L_{n,1}(\theta_1) L_{2,1}(\theta_1) + L_{n,1}(\theta_1) \frac{\mathrm{d}}{\mathrm{d}\theta_2} R_{12}(\theta_1, \theta_2)\bigg|_{\theta_2=0}.\end{aligned} \tag{5.17}$$

Writing $1 \to a, n \to n+1, 2 \to n, \theta_1 \to \theta, \theta_2 \to \tilde{\theta}$ in (5.17), we have

$$[H_{n+1,n}, L_{n+1,a}(\theta) L_{n,a}(\theta)] = \frac{\mathrm{d}}{\mathrm{d}\tilde{\theta}} R_{a,n+1}(\theta, \tilde{\theta})\bigg|_{\tilde{\theta}=0} L_{n,a}(\theta) - L_{n+1,a}(\theta) \frac{\mathrm{d}}{\mathrm{d}\tilde{\theta}} R_{a,n}(\theta, \tilde{\theta})\bigg|_{\tilde{\theta}=0}. \tag{5.18}$$

Comparing (5.18) with (4.30), we see that the operator $Q_{n,a}(\theta)$ in the Sutherland equation (4.30) is given as

$$Q_{n,a}(\theta) = \frac{\mathrm{d}}{\mathrm{d}\tilde{\theta}} R_{a,n}(\theta, \tilde{\theta})\bigg|_{\tilde{\theta}=0}. \tag{5.19}$$

6. Lax representation

Consider the operator version of an auxiliary linear problem,

$$\Psi_{n+1} = L_n \Psi_n \qquad \frac{\mathrm{d}\Psi_n}{\mathrm{d}t} = M_n \Psi_n. \tag{6.1}$$

Consistency condition for (6.1) yields the Lax equation,

$$\frac{\mathrm{d}L_n}{\mathrm{d}t} = M_{n+1} L_n - L_n M_n. \tag{6.2}$$

A model is said to be completely integrable if we can find a Lax pair, L_n and M_n, such that the Lax equation (6.2) is equivalent to the equation of motion of the model. The explicit forms of the Lax pair for some lattice integrable systems were calculated directly [42–44]. On the other hand, Izergin and Korepin [15, 45] proved that the Lax equation follows from the Yang–Baxter relation. Actually, the Lax equation is equivalent to the Sutherland equation (4.30) [46, 47] (see the appendix). The Lax pair is expressed in terms of the L-operator and the R-matrix as follows:

$$
\begin{aligned}
L_n &= L_{n,a}(\theta) \\
M_n &= \mathrm{i}L_{n,a}^{-1}(\theta)\{Q_{n,a}(\theta) - [H_{n,n-1}, L_{n,a}(\theta)]\} \\
&= \mathrm{i}L_{n,a}^{-1}(\theta)\left\{\frac{\mathrm{d}}{\mathrm{d}\tilde{\theta}}R_{a,n}(\theta,\tilde{\theta})\bigg|_{\tilde{\theta}=0} - [H_{n,n-1}, L_{n,a}(\theta)]\right\}.
\end{aligned}
\tag{6.3}
$$

Substituting (4.28) and (4.29) into (6.3), we obtain the explicit expression of the Lax operator M_n for the 1D Bariev model,

$$
M_n = \mathrm{i}\begin{pmatrix} M_n^{11} & M_n^{12} & M_n^{13} & M_n^{14} \\ M_n^{21} & M_n^{22} & M_n^{23} & M_n^{24} \\ M_n^{31} & M_n^{32} & M_n^{33} & M_n^{34} \\ M_n^{41} & M_n^{42} & M_n^{43} & M_n^{44} \end{pmatrix}
\tag{6.4}
$$

where

$$
\begin{aligned}
M_n^{11} &= \frac{s_+c_-}{c_+^2}(1+U) + \frac{s_-}{c_+}(1-U) \\
&\quad +(1+t_+)\{(1+U\tau_n^z)\sigma_{n-1}^+\sigma_n^- + (1+U\sigma_{n-1}^z)\tau_{n-1}^+\tau_n^-\} \\
&\quad +(1-t_-)\{(1+U\tau_n^z)\sigma_{n-1}^-\sigma_n^+ + (1+U\sigma_{n-1}^z)\tau_{n-1}^-\tau_n^+\} \\
M_n^{44} &= \frac{s_+}{c_-}(1+U) + \frac{s_-c_+}{c_-^2}(1-U) \\
&\quad +(1+t_-)\{(1+U\tau_n^z)\sigma_n^+\sigma_{n-1}^- + (1+U\sigma_{n-1}^z)\tau_{n-1}^-\tau_n^+\} \\
&\quad +(1-t_-)\{(1+U\tau_n^z)\sigma_n^-\sigma_{n-1}^+ + (1+U\sigma_{n-1}^z)\tau_{n-1}^+\tau_n^-\} \\
M_n^{22} &= \frac{s_+}{c_-}(1+U) + \frac{s_-}{c_+}(1-U) \\
&\quad +(1+t_-)(1+U\tau_n^z)\sigma_{n-1}^+\sigma_n^- + (1-t_-)(1+U\tau_n^z)\sigma_{n-1}^-\sigma_n^+ \\
&\quad +(1+t_+)(1+U\sigma_{n-1}^z)\tau_{n-1}^-\tau_n^+ + (1-t_+)(1+U\sigma_{n-1}^z)\tau_{n-1}^+\tau_n^- \\
M_n^{33} &= \frac{s_+}{c_-}(1+U) + \frac{s_-}{c_+}(1-U) \\
&\quad +(1-t_+)(1+U\tau_n^z)\sigma_{n-1}^+\sigma_n^- + (1+t_+)(1+U\tau_n^z)\sigma_{n-1}^-\sigma_n^+ \\
&\quad +(1-t_-)(1+U\sigma_{n-1}^z)\tau_{n-1}^-\tau_n^+ + (1+t_-)(1+U\sigma_{n-1}^z)\tau_{n-1}^+\tau_n^- \\
M_n^{14} &= M_n^{41} = 0 \\
M_n^{23} &= \frac{1}{c_+^3c_-}\{s_+c_+(1+U) - s_-c_-(1-U)\} \\
M_n^{32} &= \frac{1}{c_-^3c_+}\{s_-c_-(1-U) - s_+c_+(1+U)\} \\
M_n^{21} &= -\frac{c_-}{c_+^2}(1+U)\tau_n^+ - \frac{1}{c_+}(1+U\sigma_{n-1}^z)\tau_{n-1}^+ \\
M_n^{12} &= -\frac{1}{c_-}(1+U)\tau_n^- - \frac{1}{c_+}(1+U\sigma_{n-1}^z)\tau_{n-1}^-
\end{aligned}
$$

$$
\begin{aligned}
M_n^{31} &= -\frac{1}{c_+}(1+U\tau_n^z)\sigma_n^+ - \frac{1}{c_-}(1+U)\sigma_{n-1}^+ \\
M_n^{13} &= -\frac{1}{c_+}(1+U\tau_n^z)\sigma_n^- - \frac{c_-}{c_+^2}(1+U)\sigma_{n-1}^- \\
M_n^{42} &= -\frac{1}{c_-}(1+U\tau_n^z)\sigma_n^+ - \frac{c_+}{c_-^2}(1-U)\sigma_{n-1}^+ \\
M_n^{24} &= \frac{1}{c_-}(1+U\tau_n^z)\sigma_n^- - \frac{1}{c_+}(1-U)\sigma_{n-1}^- \\
M_n^{43} &= -\frac{1}{c_+}(1-U)\tau_n^+ - \frac{1}{c_-}(1+U\sigma_{n-1}^z)\tau_{n-1}^+ \\
M_n^{34} &= -\frac{c_+}{c_-^2}(1-U)\tau_n^- - \frac{1}{c_-}(1+U\sigma_{n-1}^z)\tau_{n-1}^-.
\end{aligned}
\tag{6.5}
$$

Here

$$c_\pm = \cos(\theta \pm \psi) \qquad s_\pm = \sin(\theta \pm \psi) \qquad t_\pm = \tan(\theta \pm \psi). \tag{6.6}$$

7. Fermionic formulation

Using the method developed by Olmedilla *et al* [21], we can formulate the Yang–Baxter relation for the 1D Bariev model in a fermionic fashion.

We write the Jordan–Wigner transformation (2.4) in a matrix form,

$$\begin{pmatrix} \sigma_n^+ \\ \sigma_n^- \end{pmatrix} = V_{n\uparrow}^2 \begin{pmatrix} c_{n\uparrow}^\dagger \\ c_{n\uparrow} \end{pmatrix} \qquad \begin{pmatrix} \tau_n^+ \\ \tau_n^- \end{pmatrix} = V_{n\downarrow}^2 \begin{pmatrix} c_{n\downarrow}^\dagger \\ c_{n\downarrow} \end{pmatrix}. \tag{7.1}$$

Here

$$
\begin{aligned}
V_{n\uparrow} &= \begin{pmatrix} v_{n\uparrow} & 0 \\ 0 & v_{n\uparrow}^{-1} \end{pmatrix} \\
V_{n\downarrow} &= \begin{pmatrix} v_{n\uparrow}u_{n\uparrow}r_n v_{n\downarrow} & 0 \\ 0 & (v_{n\uparrow}u_{n\uparrow}r_n v_{n\downarrow})^{-1} \end{pmatrix}
\end{aligned}
\tag{7.2}
$$

with the definitions

$$
\begin{aligned}
v_{ns} &= \exp\left\{\mathrm{i}\frac{\pi}{2}\sum_{k=1}^{n-1}(n_{ks}-1)\right\} \qquad u_{ns} = \exp\left\{\mathrm{i}\frac{\pi}{2}(n_{ns}-1)\right\} \\
r_n &= \exp\left\{\mathrm{i}\frac{\pi}{2}\sum_{k=n+1}^{N}(n_{k\uparrow}-1)\right\}.
\end{aligned}
\tag{7.3}
$$

We introduce the matrix V_n as

$$V_n = V_{n\uparrow} \otimes V_{n\downarrow} \tag{7.4}$$

which has the relation

$$V_{n+1} = V_n(U_{n\uparrow} \otimes U_{n\downarrow}) \tag{7.5}$$

with

$$U_{ns} = \begin{pmatrix} u_{ns} & 0 \\ 0 & u_{ns}^{-1} \end{pmatrix} \qquad s = \uparrow, \downarrow . \tag{7.6}$$

The fermionic $\mathcal{L}_n$ operator is obtained from the L_n operator through the gauge transformation

$$\mathcal{L}_{n,a}(\theta) = V_{n+1}L_{n,a}(\theta)V_n^{-1}. \tag{7.7}$$

Explicitly,

$$\mathcal{L}_{n,a}(\theta) = \begin{pmatrix} -f_{n\uparrow}(\theta) f_{n\downarrow}(\theta) & -f_{n\uparrow}(\theta) c_{n\downarrow} & \mathrm{i} c_{n\uparrow} \tilde{f}_{n\downarrow}(\theta) & \mathrm{i} c_{n\uparrow} c_{n\downarrow} \\ -\mathrm{i} \tilde{f}_{n\uparrow}(\theta) c^{\dagger}_{n\downarrow} & \tilde{f}_{n\uparrow}(\theta) g_{n\downarrow}(\theta) & c_{n\uparrow} c^{\dagger}_{n\downarrow} & \mathrm{i} c_{n\uparrow} \tilde{g}_{n\downarrow}(\theta) \\ c^{\dagger}_{n\uparrow} f_{n\downarrow}(\theta) & c^{\dagger}_{n\uparrow} c_{n\downarrow} & g_{n\uparrow}(\theta) \tilde{f}_{n\downarrow}(\theta) & g_{n\uparrow}(\theta) c_{n\downarrow} \\ -\mathrm{i} c^{\dagger}_{n\uparrow} c^{\dagger}_{n\downarrow} & c_{n\uparrow} g_{n\downarrow}(\theta) & \mathrm{i} \tilde{g}_{n\uparrow}(\theta) c^{\dagger}_{n\downarrow} & -\tilde{g}_{n\uparrow}(\theta) \tilde{g}_{n\downarrow}(\theta) \end{pmatrix} \tag{7.8}$$

where with $s = \uparrow$ or $\downarrow$

$$\begin{aligned} f_{ns}(\theta) &= \sin(\theta+\psi) - \{\sin(\theta+\psi) - \mathrm{i}\cos(\theta+\psi)\} n_{n,s} \\ \tilde{f}_{ns}(\theta) &= \sin(\theta-\psi) - \{\sin(\theta-\psi) - \mathrm{i}\cos(\theta-\psi)\} n_{n,s} \\ g_{ns}(\theta) &= \cos(\theta+\psi) - \{\cos(\theta+\psi) + \mathrm{i}\sin(\theta+\psi)\} n_{n,s} \\ \tilde{g}_{ns}(\theta) &= \cos(\theta-\psi) - \{\cos(\theta-\psi) + \mathrm{i}\sin(\theta-\psi)\} n_{n,s}. \end{aligned} \tag{7.9}$$

The fermionic $\mathcal{R}$-matrix is defined by the graded Yang–Baxter relation,

$$\check{\mathcal{R}}_{12}(\theta_1,\theta_2)[\mathcal{L}_n(\theta_1)\otimes_s \mathcal{L}_n(\theta_2)] = [\mathcal{L}_n(\theta_2)\otimes_s \mathcal{L}_n(\theta_1)]\check{\mathcal{R}}_{12}(\theta_1,\theta_2) \tag{7.10}$$

where $\otimes_s$ means the Grassmann direct product

$$\begin{aligned} &[A\otimes_s B]_{\rho\gamma,\beta\delta} = (-1)^{[P(\rho)+P(\beta)]P(\gamma)} A_{\rho\beta} B_{\gamma\delta} \\ &P(1) = P(4) = 0 \qquad P(2) = P(3) = 1. \end{aligned} \tag{7.11}$$

The fermionic $\mathcal{R}$-matrix is derived from the R-matrix. Substituting the expression (7.7) into (5.1), we have

$$\begin{aligned} &\check{R}_{12}(\theta_1,\theta_2)[(V^{-1}_{n+1}\mathcal{L}_n(\theta_1)V_n)\otimes(V^{-1}_{n+1}\mathcal{L}_n(\theta_2)V_n)] \\ &\qquad = [(V^{-1}_{n+1}\mathcal{L}_n(\theta_2)V_n)\otimes(V^{-1}_{n+1}\mathcal{L}_n(\theta_1)V_n)]\check{R}_{12}(\theta_1,\theta_2). \end{aligned} \tag{7.12}$$

With the diagonal matrix

$$W = \mathrm{diag}(1, 1, -\mathrm{i}, -\mathrm{i}, -\mathrm{i}, -\mathrm{i}, 1, 1, -1, -1, \mathrm{i}, \mathrm{i}, \mathrm{i}, \mathrm{i}, -1, -1) \tag{7.13}$$

we can rewrite equation (7.12) as

$$\begin{aligned} &\check{R}_{12}(\theta_1,\theta_2)[V^{-1}_{n+1}\otimes V^{-1}_{n+1}]W^{-1}(\mathcal{L}_n(\theta_1)\otimes_s\mathcal{L}_n(\theta_2))W[V_n\otimes V_n] \\ &\qquad = [V^{-1}_{n+1}\otimes V^{-1}_{n+1}]W^{-1}(\mathcal{L}_n(\theta_2)\otimes_s\mathcal{L}_n(\theta_1))W[V_n\otimes V_n]\check{R}(\theta_1,\theta_2). \end{aligned} \tag{7.14}$$

In the derivation of (7.14) the following relations are useful:

$$\begin{aligned} &c_{n,s}u_{n,s} = c_{n,s}u^{-1}_{n,s} = c_{n,s} \qquad u_{n,s}c_{n,s} = -u^{-1}_{n,s}c_{n,s} = -ic_{n,s} \\ &u^{-1}_{n,s}c^{\dagger}_{n,s} = u_{n,s}c^{\dagger}_{n,s} = c^{\dagger}_{n,s} \qquad c^{\dagger}_{n,s}u^{-1}_{n,s} = -c^{\dagger}_{n,s}u^{-1}_{n,s} = ic^{\dagger}_{n,s}. \end{aligned} \tag{7.15}$$

Since the relations

$$\check{R}_{12}(\theta_1,\theta_2) = (V_j\otimes V_j)\check{R}_{12}(\theta_1,\theta_2)(V^{-1}_j\otimes V^{-1}_j) \tag{7.16}$$

hold for $j = n$ and $j = n+1$, the fermionic $\mathcal{R}$-matrix is obtained as

$$\check{\mathcal{R}}_{12}(\theta_1,\theta_2) = W\check{R}_{12}(\theta_1,\theta_2)W^{-1} \tag{7.17}$$

or explicitly

$$
\begin{pmatrix}
\alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_5 & 0 & 0 & \mathrm{i}\alpha_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_6 & 0 & 0 & 0 & 0 & 0 & \mathrm{i}\alpha_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_4 & 0 & 0 & -\mathrm{i}\alpha_{18} & 0 & 0 & \mathrm{i}\alpha_{15} & 0 & 0 & -\alpha_{20} & 0 & 0 & 0 \\
0 & -\mathrm{i}\alpha_{12} & 0 & 0 & \alpha_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathrm{i}\alpha_{17} & 0 & 0 & \alpha_9 & 0 & 0 & -\alpha_{19} & 0 & 0 & -\mathrm{i}\alpha_{18} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_7 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i}\alpha_{14} & 0 & 0 \\
0 & 0 & -\mathrm{i}\alpha_{12} & 0 & 0 & 0 & 0 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\mathrm{i}\alpha_{16} & 0 & 0 & -\alpha_{19} & 0 & 0 & \alpha_{10} & 0 & 0 & \mathrm{i}\alpha_{15} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_8 & 0 & 0 & -\mathrm{i}\alpha_{14} & 0 \\
0 & 0 & 0 & -\alpha_{21} & 0 & 0 & \mathrm{i}\alpha_{17} & 0 & 0 & -\mathrm{i}\alpha_{16} & 0 & 0 & \alpha_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{i}\alpha_{13} & 0 & 0 & 0 & 0 & 0 & \alpha_8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{i}\alpha_{13} & 0 & 0 & \alpha_7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_3
\end{pmatrix}
$$

The monodromy matrix $\mathcal{T}(\theta)$ is defined by

$$\mathcal{T}(\theta) = \mathcal{L}_{N,a}(\theta) \ldots \mathcal{L}_{1,a}(\theta) = \prod_{n=1}^{\overleftarrow{N}} \mathcal{L}_{n,a}(\theta). \tag{7.18}$$

From the local graded Yang–Baxter relation (7.10), we have the global graded Yang–Baxter relation for $\mathcal{T}(\theta)$

$$\mathcal{R}_{12}(\theta_1, \theta_2)[\mathcal{T}(\theta_1) \otimes_s \mathcal{T}(\theta_2)] = [\mathcal{T}(\theta_2) \otimes_s \mathcal{T}(\theta_1)]\mathcal{R}_{12}(\theta_1, \theta_2). \tag{7.19}$$

The transfer matrix is defined by the supertrace of the monodromy matrix,

$$T(\theta) = \mathrm{str}\mathcal{T}(\theta) \equiv \mathrm{tr}\{(\sigma^z \otimes \sigma^z)\mathcal{T}(\theta)\}. \tag{7.20}$$

From the global graded Yang–Baxter relation (7.19), we obtain the commutativity of the transfer matrices

$$[T(\theta), T(\theta')] = 0. \tag{7.21}$$

8. Concluding remarks

In this paper, we have studied the exact integrability of the 1D Bariev model in the framework of the QISM. We have found the higher conserved currents, which enables us to assume the form of the L-operator. We have also found the corresponding R-matrix. The R-matrix does not have the 'difference property' (3.3) for the spectral parameter. This is consistent with the fact that the Hamiltonian density of the 1D Bariev model does not satisfy the Reshetikhin's criterion. We conjecture that the R-matrix (5.4) is a solution of the Yang–Baxter equation

$$R_{12}(\theta_1, \theta_2)R_{13}(\theta_1, \theta_3)R_{23}(\theta_2, \theta_3) = R_{23}(\theta_2, \theta_3)R_{13}(\theta_1, \theta_3)R_{12}(\theta_1, \theta_2) \tag{8.1}$$

under the constraint

$$\frac{\sin 2\psi_1}{\sin 2\theta_1} = \frac{\sin 2\psi_2}{\sin 2\theta_2} = \frac{\sin 2\psi_3}{\sin 2\theta_3} = U. \tag{8.2}$$

Indeed we have confirmed some non-trivial matrix elements of the Yang–Baxter equation (8.1). The Yang–Baxter equation (8.1) implies that a more general inhomogeneous model is integrable with a transfer matrix

$$T(\theta, \{\theta_n\}) = \mathrm{tr}_a \overleftarrow{\prod_n} R_{a,n}(\theta, \theta_n) \tag{8.3}$$

such that

$$[T(\theta, \{\theta_n\}), T(\theta^{'}, \{\theta_n\})] = 0 \tag{8.4}$$

where the parameters $\{\theta_n\}$ are arbitrary constants.

The constraint (8.2) is very similar to the one under which the R-matrix of the 1D Hubbard model satisfies the Yang–Baxter equation [23]. The R-matrix of the 1D Hubbard model is expressed in a compact form [20, 23]. It is desirable to find a compact expression for the R-matrix (5.4).

One of the problems which we did not discuss in this paper is the eigenvalues of the transfer matrix. It may be possible to apply the diagonal-to-diagonal Bethe ansatz method and obtain the free energy of the corresponding two-layer vertex model [48, 49]. We shall report this problem elsewhere. The recent approach of the algebraic Bethe ansatz for the 1D Hubbard model [24] may also be applied to the present model.

More recently, the 1D Bariev model was generalized to a two-parameter correlated hopping model [29]. This model is also solvable by the coordinate Bethe ansatz and contains several other models as special cases [29]. It is shown that the underlying R-matrix comes from the four-dimensional representation of the quantum superalgebra $U_q(gl(2|1))$ [50]. Surprisingly, this R-matrix has a spectral parameter which satisfies the 'difference property' (3.3). This fact seems to be contradictory to our result. However, in their argument [50], the 1D Bariev model corresponds to the extreme case where a free-parameter in the representation of $U_q(gl(2|1))$ diverges, and it is not straightforward to reproduce the R-matrix of the 1D Bariev model. We shall try to clarify these relationships in the future.

Finally, we would like to mention a different approach to the integrability of the 1D Bariev model, based on the representation of the affine Hecke algebra [51]. It is shown that the 1D Bariev model becomes the δ-function interacting fermions in the continuum limit.

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Appendix. Lax equation from the Sutherland equation (4.30)

From (4.30), we can derive the following two relations:

$$\begin{aligned}[H_{n+1,n}, L_{n,a}(\theta)] = &-Q_{n,a}(\theta) + L^{-1}_{n+1,a}(\theta)Q_{n+1,a}(\theta)L_{n,a}(\theta) \\ &-L^{-1}_{n+1,a}(\theta)[H_{n+1,n}, L_{n+1,a}(\theta)]L_{n,a}(\theta)\end{aligned} \tag{A.1}$$

$$[H_{n,n-1}, L_{n,a}(\theta)] = Q_{n,a}(\theta) - L_{n,a}(\theta)Q_{n-1,a}(\theta)L^{-1}_{n-1,a}(\theta) \\ -L_{n,a}(\theta)[H_{n,n-1}, L_{n-1,a}(\theta)]L^{-1}_{n-1,a}(\theta). \tag{A.2}$$

Then the time evolution of the L-operator becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}L_{n,a}(\theta) &= \mathrm{i}[H, L_{n,a}(\theta)] \\ &= \mathrm{i}[H_{n+1,n}, L_{n,a}(\theta)] + \mathrm{i}[H_{n,n-1}, L_{n,a}(\theta)] \\ &= \mathrm{i}L^{-1}_{n+1,a}(\theta)\{Q_{n+1,a}(\theta) - [H_{n+1,n}, L_{n+1,a}]\}L_{n,a}(\theta) \\ &\quad -\mathrm{i}L_{n,a}(\theta)\{Q_{n-1,a}(\theta) + [H_{n,n-1}, L_{n-1,a}]\}L^{-1}_{n-1,a}(\theta).\end{aligned} \tag{A.3}$$

Again from (4.30), the following relation holds:

$$L^{-1}_{n,a}(\theta)\{Q_{n,a}(\theta) - [H_{n,n-1}, L_{n,a}(\theta)]\} = \{Q_{n-1,a}(\theta) + [H_{n,n-1}, L_{n-1,a}(\theta)]\}L^{-1}_{n-1,a}(\theta). \tag{A.4}$$

Thus equation (A.3) is cast into the form of the Lax equation (6.2),

$$\frac{\mathrm{d}}{\mathrm{d}t}L_{n,a}(\theta) = M_{n+1,a}(\theta)L_{n,a}(\theta) - L_{n,a}(\theta)M_{n,a}(\theta) \tag{A.5}$$

where

$$M_{n,a}(\theta) = \mathrm{i}L^{-1}_{n,a}(\theta)\{Q_{n,a}(\theta) - [H_{n,n-1}, L_{n,a}(\theta)]\}. \tag{A.6}$$

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